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Generalized 'useful' AG and 'useful' JS-Divergence Measures and their Bounds

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Abstract

In the present paper we consider one parametric generalization of 'useful' AG and 'useful' JS-divergence measures. Both the generalizations can be expressed as particular cases of Csiszar f-divergence. Under some conditions of probability distributions, some bounds inequalities are also obtained.

Keywords:

Csiszar's f-divergence;
Useful' Relative information;
KL Divergence;
Inequalities;
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1. Introduction

Let $\Delta_n^+ = \left\{ p = (p_1, p_2, ..., p_n), p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}$, be a set of all possible discrete probability distributions of a random variable X having utility distribution $U = \{(u_1, u_2, ..., u_n); u_i > 0 \text{ for all } i\}$ attached to each $P \in \Delta_n^+$ such that $u_i > 0$ is utility of an event having probability of occurrence $p_i > 0$.

The following measure of 'useful' directed divergence or 'useful' relative information is given by:

$$I(P;Q;U) = \frac{\sum_{i=1}^{n} u_i p_i \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^{n} u_i p_i} \tag{1}$$

It may be noted that (1) is a generalization of the measure

$$H(P;U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} \tag{2}$$

It can be observed that (1) is not symmetric in P and Q and its symmetric version is given by

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$$J(P;Q;U) = I(P;Q;U) + I(Q;P;U)$$

$$= \frac{\sum_{i=1}^{n} u_i (p_i - q_i) \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^{n} u_i p_i}, \tag{3}$$

Where $\sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i$

When utilities are ignored or $u_i = 1$ for each i, (3) reduces to

$$J(P;Q) = \sum_{i=1}^{n} (p_i - q_i) \log\left(\frac{p_i}{q_i}\right), \tag{4}$$

Which is Jeffrey's divergence measure and it is written as J-divergence. In the next section we give some generalized 'Useful' Symmetric Divergence Measures.

2. Generalized 'Useful' Symmetric Divergence Measures

'Useful' Hellinger Discrimination

$$h(P;Q;U) = 1 - B(P;Q;U) = \frac{1}{2} \frac{\sum_{i=1}^{n} u_i (p_i^{1/2} - q_i^{1/2})^2}{\sum_{i=1}^{n} u_i p_i} \tag{5}$$

Where

$$B(P;Q;U) = \frac{\sum_{i=1}^{n} u_i \sqrt{p_i q_i}}{\sum_{i=1}^{n} u_i p_i} \tag{6}$$

is the well-known Bhattacharyya [1] coefficient.

'Useful' Triangular Discrimination

$$\Delta(P;Q;U) = 2[1 - W(P;Q;U)] = \frac{\sum_{i=1}^{n} u_i \frac{(p_i - q_i)^2}{p_i + q_i}}{\sum_{i=1}^{n} u_i p_i} \tag{7}$$

Where

$$W(P;Q;U) = \frac{\sum_{i=1}^{n} u_i \frac{2 p_i q_i}{p_i + q_i},}{\sum_{i=1}^{n} u_i p_i} \tag{8}$$

is the well-known harmonic mean divergence.

'Useful' Symmetric Chi-square Divergence

$$\psi(P;Q;U) = \chi^2(P;Q;U) + \chi^2(P;Q;U) = \frac{\sum_{i=1}^{n} u_i \frac{(p_i - q_i)^2 (p_i + q_i)}{p_i q_i}}{\sum_{i=1}^{n} u_i p_i} \tag{9}$$

Where

$$\chi^2(P;Q;U) = \frac{\sum_{i=1}^{n} u_i \frac{(p_i - q_i)^2}{q_i}}{\sum_{i=1}^{n} u_i p_i} = \frac{\sum_{i=1}^{n} u_i \frac{p_i^2}{q_i}}{\sum_{i=1}^{n} u_i p_i} - 1, \tag{10}$$

is the well-known 'useful' $\chi^2 -$ divergence.

'Useful' J-Divergence

$$F(P;Q;U) = \frac{\sum_{i=1}^{n} u_i (p_i - q_i) \log \left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^{n} u_i p_i} \tag{11}$$

'Useful' Jensen-Shannon Divergence

$$K(P;Q;U) = \frac{1}{2}\left[\frac{\sum_{i=1}^{n} u_i p_i \log\left(\frac{2p_i}{p_i+q_i}\right)}{\sum_{i=1}^{n} u_i p_i} + \frac{\sum_{i=1}^{n} u_i q_i \log\left(\frac{2q_i}{p_i+q_i}\right)}{\sum_{i=1}^{n} u_i q_i}\right] \tag{12}$$

'Useful' Arithmetic – Geometric Divergence

$$J(P;Q;U) = \frac{\sum_{i=1}^{n} u_i \left(\frac{p_i+q_i}{2}\right) \log\left(\frac{p_i+q_i}{2\sqrt{p_i q_i}}\right)}{\sum_{i=1}^{n} u_i p_i} \tag{13}$$

After simplification, we can write

$$F(P;Q;U) = 4\,[K(P;Q;U) + J(P;Q;U)] \tag{14}$$

The measures $F(P;Q;U)$, $K(P;Q;U)$ and $J(P;Q;U)$ can also be written as

$$F(P;Q;U) = L(P;Q;U) + L(Q;P;U) \tag{15}$$

$$K(P;Q;U) = \frac{1}{2}\left[L\left(P;\frac{P+Q}{2};U\right) + L\left(Q;\frac{P+Q}{2};U\right)\right] \tag{16}$$

and

$$J(P;Q;U) = \frac{1}{2}\left[L\left(\frac{P+Q}{2};P;U\right) + L\left(\frac{P+Q}{2};Q;U\right)\right] \tag{17}$$

Where

$$L(P;Q;U) = \frac{\sum_{i=1}^{n} u_i p_i \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^{n} u_i p_i} \tag{18}$$

is the well known Kullback Leibler 'useful' relative information.

The measure (13) is also known by 'useful' Jensen difference divergence measure. The measure (14) is new in the literature and is called as 'useful' arithmetic- geometric mean divergence measure. It is the generalization of the measure which is studied for the first time by Taneja [4] We call the measures given in (5), (7), (9), (11), (13) and (14) by 'useful' symmetric divergence measures, since they are 'useful' symmetric with respect to the probability distributions P and Q and utility distribution U. While the measures (10) and (18) are not symmetric with respect to probability distributions.

When utilities are ignored all the above 'useful' divergence measures reduced to divergence measures given by Taneja [9].

3. Unified 'Useful' AG and JS – Divergence of Type s

Here first of all we will present the genelization of 'useful' Kullback-Leibler's divergence measures

'Useful' Relative Information of Type s

$$\Phi_s(P;Q;U)=\begin{cases} L_s(P;Q;U)=[s(s-1)]^{-1}\left[\dfrac{\sum_{i=1}^{n}u_i p_i^s q_i^{1-s}}{\sum_{i=1}^{n}u_i p_i}-1\right], & s\neq 0,1 \\ L(Q;P;U)=\dfrac{\sum_{i=1}^{n}u_i q_i \log\left(\dfrac{q_i}{p_i}\right)}{\sum_{i=1}^{n}u_i p_i}, & s=0 \\ L(P;Q;U)=\dfrac{\sum_{i=1}^{n}u_i p_i \log\left(\dfrac{p_i}{q_i}\right)}{\sum_{i=1}^{n}u_i p_i}, & s=1 \end{cases} \tag{19}$$

for all $s\in R$

The measure (19) is due to Cressie and Read [2]. For more details on this measure refer to Taneja [5] and Taneja and Kumar [10, 3] and reference therein.

The measure (19) gives the following particular cases:

(i) $\Phi_{-1}(P;Q;U)=\frac{1}{2}\chi^2(Q;P;U)$

(ii) $\Phi_0(P;Q;U)=L(Q;P;U)$

(iii) $\Phi_{1/2}(P;Q;U)=4[1-B(P;Q;U)]=4h(P;Q;U)$

(iv) $\Phi_1(P;Q;U)=L(P;Q;U)$

(v) $\Phi_2(P;Q;U)=\frac{1}{2}\chi^2(P;Q;U)$

Here, we observe that $\Phi_2(P;Q;U)=\Phi_{-1}(Q;P;U)$ and $\Phi_1(P;Q;U)=\Phi_0(Q;P;U)$

Provided $\sum_{i=1}^{n}u_i p_i=\sum_{i=1}^{n}u_i q_i$

3.1. 'Useful' J-Divergence of Type s. Replace $L(P;Q;U)$ by $\Phi_s(P;Q;U)$ in the relation (15), we get

$$V_s(P;Q;U)=\Phi_s(P;Q;U)+\Phi_s(P;Q;U)$$

$$=\begin{cases} F_s(P;Q;U)=[s(s-1)]^{-1}\left[\dfrac{\sum_{i=1}^{n}u_i(p_i^s q_i^{1-s}+p_i^{1-s}q_i^s)}{\sum_{i=1}^{n}u_i p_i}-2\right], & s\neq 0,1 \\ F(P;Q;U)=\dfrac{\sum_{i=1}^{n}u_i(p_i-q_i)\log\left(\dfrac{p_i}{q_i}\right)}{\sum_{i=1}^{n}u_i p_i} & s=0,1 \end{cases} \tag{20}$$

The expression (20) admits the following particular cases:

(i) $V_{-1}(P;Q;U)=V_2(P;Q;U)=\frac{1}{2}\Psi(P;Q;U)$

(ii) $V_0(P;Q;U)=V_1(P;Q;U)=F(P;Q;U)$

(iii) $V_{1/2}(P;Q;U)=8h(P;Q;U)$

Provided $\sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i$

Remark 1. The expression (20) is the modified form of the measure already known in the literature:

$$V_s^1(P;Q;U) = \begin{cases} F_s(P;Q;U) = (s-1)^{-1}\left[\dfrac{\sum_{i=1}^{n} u_i\left(p_i^s q_i^{1-s} + p_i^{1-s} q_i^s\right)}{\sum_{i=1}^{n} u_i p_i} - 2\right], & s \neq 1,\ s > 0 \\ F(P;Q;U) = \dfrac{\sum_{i=1}^{n} u_i (p_i - q_i) \log\left(\dfrac{p_i}{q_i}\right)}{\sum_{i=1}^{n} u_i p_i}, & s = 1 \end{cases} \tag{21}$$

3.2. 'Useful' AG and 'Useful' JS – Divergence Measures of type s. Replace $L(P;Q;U)$ by $\Phi_s(P;Q;U)$ in the relation (17) then we have a unified generalization of the 'Useful' AG and 'Useful' JS - Divergence which is given by

$$W_s(P;Q;U) = \frac{1}{2}\left[\Phi_s\left(\frac{P+Q}{2};P;U\right) + \Phi_s\left(\frac{P+Q}{2};Q;U\right)\right]$$

$$= \begin{cases} KJ_s(P;Q;U) = [s(s-1)]^{-1}\left[\dfrac{\sum_{i=1}^{n} u_i\left(\dfrac{p_i^{1-s}+q_i^{1-s}}{2}\right)\left(\dfrac{p_i+q_i}{2}\right)^s}{\sum_{i=1}^{n} u_i p_i} - 1\right], & s \neq 0,1 \\ K(P;Q;U) = \dfrac{1}{2}\left[\dfrac{\sum_{i=1}^{n} u_i p_i \log\left(\dfrac{2p_i}{p_i+q_i}\right)}{\sum_{i=1}^{n} u_i p_i} + \dfrac{\sum_{i=1}^{n} u_i q_i \log\left(\dfrac{2q_i}{p_i+q_i}\right)}{\sum_{i=1}^{n} u_i q_i}\right] & s = 0 \\ J(P;Q;U) = \dfrac{\sum_{i=1}^{n} u_i\left(\dfrac{p_i+q_i}{2}\right)\log\left(\dfrac{p_i+q_i}{2\sqrt{p_i q_i}}\right)}{\sum_{i=1}^{n} u_i p_i} & s = 1 \end{cases} \tag{22}$$

The measure (22) gives the following particular cases:

(i) $W_{-1}(P;Q;U) = \frac{1}{4}\Delta(P;Q;U)$

(ii) $W_0(P;Q;U) = K(P;Q;U)$

(iii) $W_{1/2}(P;Q;U) = 4d(P;Q;U)$

(iv) $W_1(P;Q;U) = J(P;Q;U)$

(v) $W_2(P;Q;U) = \frac{1}{16}\Psi(P;Q;U)$

Provided $\sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i$

The 'useful' measure $d(P;Q;U)$ given in part (iii) is not studied elsewhere and given by

$$d(P;Q;U)=1-\frac{\sum_{i=1}^{n}u_i\left(\frac{\sqrt{p_i}+\sqrt{q_i}}{2}\right)\left(\frac{\sqrt{p_i}+\sqrt{q_i}}{2}\right)}{\sum_{i=1}^{n}u_i p_i} \tag{23}$$

We can also write

$$W_{1-s}(P;Q;U)=\frac{1}{2}\left[\Phi_s\left(P;\frac{P+Q}{2};U\right)+\Phi_s\left(Q;\frac{P+Q}{2};U\right)\right] \tag{24}$$

Thus we have two 'useful' symmetric divergences of type a given by (20) and (21) generalizing the six 'useful' symmetric divergence measures given in Section 2. In this paper our aim is to study the bounds on 'useful' AG and 'useful' JS divergences of type s and to find inequalities among them. These studies we shall do by making use of the properties of 'Useful' Csiszar's *f*-divergence.

4. Bounds on 'Useful' AG and 'Useful' JS – Divergence Measures of type s

Firstly we shall give two important properties of AG and JS- divergences of type s.

Property 1. The measure $W_s(P;Q;U)$ is nonnegative and convex in the pair of probability distributions $(P,Q)\in\Delta^+_n\times\Delta^+_n$ for all $s\in(-\infty,\infty)$,

Proof. For all $x>0$ and $s\in(-\infty,\infty)$ let us consider in (16)

$$\psi_s(x)=\begin{cases}[s(s-1)]^{-1}\left[\left(\frac{x^{1-s}+1}{2}\right)\left(\frac{x+1}{2}\right)^s-\left(\frac{x+1}{2}\right)\right], & s\neq 0,1\\ \frac{x}{2}\,In\,x-\left(\frac{x+1}{2}\right)In\left(\frac{x+1}{2}\right) & s=0,\\ \left(\frac{x+1}{2}\right)In\left(\frac{x+1}{2\sqrt{x}}\right) & s=1\end{cases} \tag{25}$$

then we have $C_f(P\|Q)=W_s(P;Q;U)$, Where $W_s(P;Q;U)$ is as given by (19).

Moreover,

$$\psi_s'(x)=\begin{cases}(s-1)^{-1}\left[\frac{1}{8}\left[\left(\frac{x+1}{2x}\right)^s-1\right]-\frac{x^{-s}-1}{4}\left(\frac{x+1}{2}\right)^{s-1}\right] & s\neq 0,1\\ -\frac{1}{2}\,ln\left(\frac{x+1}{2x}\right), & s=0,\\ 1-x^{-1}-\,ln\,x-2ln\left(\frac{2}{x+1}\right), & s=1\end{cases} \tag{26}$$

And

$$\psi''_s=\left(\frac{x^{-s-1}+1}{8}\right)\left(\frac{x+1}{2}\right)^{s-2} \tag{27}$$

Thus we have $\psi_s''(x)>0$ for all $x>0$, and hence, $\psi_s(x)$ is convex for all $x>0$, Also we have $\psi_s(1)=0$. In view of this we can say that 'useful' AG and 'useful' JS – divergences of type s is nonnegative and convex in the pair of probability distributions $(P;Q)\in\Delta^+_n\times\Delta^+_n$.

Property 2. The measure $W_s(P;Q;U)$ is monotonically increasing in s for all $s\geq -1$.

Proof. Let us consider the first order derivative of (25) with respect to s.

$$\begin{aligned}m_s(x)&=\frac{d}{ds}(\psi(x)) \quad (28)\\ &=[s(s-1)]^{-2}\left(\frac{x+1}{2}\right)^s[(2s-1)(x^{1-s}+x+2)\\ &\quad -s(s-1)\,(x^{1-s}+1)\,ln\left(\frac{x+1}{2}\right)\Big], \quad s\neq 0,1\end{aligned}$$

Now, calculating the first and second order derivatives of (27) with respect to x , we get

$$m_s'(x) = \frac{1-2s}{s^2(1-s)^2}[x^2 + x^{1-s} - (x+1)] + \frac{1}{s(s-1)}(x^s - x^{1-s})\ ln\ x,\ s \neq 0, 1$$

and

$$m_s''(x) = \frac{1}{2x^2(x+1)^2}\left(\frac{x+1}{2}\right)^2 \left[x^{1-s}\ ln\left(\frac{x+1}{2x}\right) + x^2\ ln\left(\frac{x+1}{2}\right)\right] \tag{29}$$

respectively.

Since $(x-1)^2 \geq 0$ for any x, this give us

$$\ln\left(\frac{x+1}{2x}\right) \geq ln\left(\frac{2}{x+1}\right). \tag{30}$$

Now for all $0 < x \leq 1$ and for any $s \geq -1$, we have $x^{1-s} \geq x^2$. this together with (30) gives

$$m_s''(x) \geq 0,\ \text{For all } 0 < x \leq 1 \text{ and } s \geq -1. \tag{31}$$

Reorganizing (29), we can write

$$m_s''(x) \frac{x^{1-s}}{2x^2(x+1)^2}\left(\frac{x+1}{2}\right)^s \left[\left(x^{1+s}+1\right) ln\left(\frac{x+1}{2}\right) - ln\ x\right]$$

Again for all $x \geq 1$, and $s \geq -1$ we have $x^{1+s} + 1 \geq 2$. This gives

$$(x^{1+s}+1)\ln\left(\frac{x+1}{2}\right) \geq 2\ ln\left(\frac{x+1}{2}\right) \geq ln\ x, \tag{32}$$

where we have used the fact that $(x+1)^2 \geq 4x$ for any x.

In view of (32), we have

$$m_s''(x) \geq 0,\ \text{For all } x \geq 1 \text{ and } s \geq -1. \tag{33}$$

Combining (31) and (33), we have

$$m_s''(x) \geq 0,\ \text{for all } x > 0 \text{ and } s \geq -1. \tag{34}$$

Since $m_s(1) = m_s'(1) = 0$, then (34) together with Lemma 2 complete the required proof.

By taking $s = -1, 0\frac{1}{2}, 1\ and\ 2,$ and applying property 4, one gets

$$\frac{1}{4}\Delta(P;Q;U) \leq S(P;Q;U) \leq 4d(P;Q;U) \leq G(P;Q;U) \leq \frac{1}{16}\Psi(P;Q:U) \tag{35}$$

Theorem 1. The following bounds holds:

$$0 \leq W_s(P;Q;U) \leq Fw_s(P;Q;U) \leq Bw_s(r, R), \tag{36}$$

$$0 \leq W_s(P;Q;U) \leq Aw_s(r, R) \leq Bw_s(r, R), \tag{37}$$

$$\left|W_s(P;Q;U) - \frac{1}{2}Fw_s(P;Q;U)\right| \tag{38}$$

$$\leq \min\left\{\frac{1}{8}\delta w_s(r, R)\chi^2(P;Q;U),\ \frac{1}{12}\|\psi_s'''\|_\infty |\chi|^3(P;Q;U),\ \overset{R}{\underset{r}{V}}(v_s')V(P;Q;U)\right\},$$

and

$$|W_s(P;Q;U) - F^*w_s(P;Q;U) \tag{39}$$

$$\leq \min\left\{\frac{1}{8}\delta w_s(r, R)X^2(P;Q;U),\ \frac{1}{24}\|\psi_s'''\|_\infty |X|^3(P;Q;U),\ \frac{1}{2}\overset{R}{\underset{r}{V}}(v_s')V(P;Q;U)\right\}$$

Where

$$Fw_s(P;Q;U) = \begin{cases} \dfrac{\frac{1}{2}\sum_i^n u_i(p_i-q_i)\left\{(s-1)^{-1}\left(\frac{p_i^{1-s}+q_i^{1-s}}{2}\right)\left(\frac{p_i+q_i}{2p_i}\right)^2\right\}}{\sum_{i=1}^n u_i p_i} & s\neq 0,1 \\ J\left(\frac{P+Q}{2};P;U\right), & s=0 \\ \frac{1}{4}\left[\chi^2(P;Q:U)-J(P;Q:U)\right]+J\left(\frac{P+Q}{2};Q;U\right) & s=1 \end{cases} \tag{40}$$

$$F^*w_s(P;Q:U) = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{s+1}\sum_{i=1}^n u_i(p_i-q_i)\left\{(s-1)^{-1}\left[\left(\frac{p_i+3q_i}{p_i+q_i}\right)^{s-1}+\left(\frac{p_i+3q_i}{2q_i}\right)^{s-1}\right]-s^{-1}\left(\frac{p_i+3q_i}{2q_i}\right)^{s}\right\}}{\sum_{i=1}^n u_i p_i}, & s\neq 0,1 \\ 2J\left(\frac{P+Q}{2};\frac{P+3Q}{4};U\right), & s=0 \\ \frac{1}{4}\Delta(P;Q;U)-\frac{1}{2}J\left(\frac{P+Q}{2};Q;U\right)+2J\left(\frac{P+3Q}{4};Q;U\right), & s=1 \end{cases} \tag{41}$$

$$Bw_s(r,R)=\frac{(R-r)^2}{16}\left[\frac{1}{rR}L_{s-1}^{s-1}\left(\frac{r+1}{2r},\frac{R+1}{2R}\right) -\frac{1}{2rR}L_{s-2}^{s-2}\left(\frac{r+1}{2r},\frac{R+1}{2r}\right)+\frac{1}{2}L_{s-2}^{s-2}\left(\frac{r+1}{2},\frac{R+1}{2}\right)\right] \tag{42}$$

$$Av_s(r,R) = \begin{cases} [s(s-1)]^{-1}\left\{\frac{1}{R-r}\left[(1-r)\left(\frac{R^{1-s}+1}{2}\right)\left(\frac{R+1}{2}\right)^s +(R+1)\left(\frac{r^{1-s}+1}{2}\right)\left(\frac{r+1}{2}\right)^s\right]-1\right\} & s\neq 0,1 \\ \frac{1}{2(R-r)}\left\{(1-r)\left[R\ ln\ R-(1-R)\ ln\ \frac{R+1}{2}\right] +(R-1)\left[r\ In\ r-(r+1)\ In\ \left(\frac{r+1}{2}\right)\right]\right\}, & s=0 \\ \frac{1}{2}\left\{(1-rR)L_{-1}^{-1}(r+1,R+1)+In\left[\frac{(r+1)(R+1)}{8}\right]\right\}, & s=1 \end{cases} \tag{43}$$

$$\delta w_s(r,\ R)=\left(\frac{r^{-s-1}+1}{8}\right)\left(\frac{r+1}{2}\right)^{s-2} -\left(\frac{R^{-s-1}+1}{8}\right)\left(\frac{R+1}{2}\right)^{s-2},\quad -1\leq s\leq 2, \tag{44}$$

$$\|\psi_s'''\|\infty=\frac{1}{2(r^3+1)}\left(\frac{r+1}{2}\right)^2\times \times\left[3r^{-s-1}+(S+1)r^{--2}+(2-s)\right],\ -1\geq s\leq 2, \tag{45}$$

and

$$\overset{R}{\underset{r}{V}}(\psi') = \frac{4}{R-r} Bw_s(r,\ R) \tag{46}$$

Proof. By making some calculations and applying Theorem 2 we get the inequalities (36) and (37).Now, we shall prove the inequalities (38) and (39). The third order derivative of the function $\psi_s(x)$ is given by

$$\psi_s'''(x) = -\frac{1}{2(x+1)^3}\left(\frac{x+1}{2}\right)^s \times$$
$$\times[3x^{-s-1} + (s+1)x^{-2-s} + (2-s)],\ \ x \in (0,\ \infty) \tag{47}$$

This gives

$$\psi_s'''(x) \le 0,\ \ -1 \le s \le 2. \tag{48}$$

From (48), we can say that the function $\phi''(s)$ is monotonically decreasing in $x \in (0,\ \infty)$, and hence. for all $x \in [r, R]$, we have

$$\delta w_s(r,R) = \psi''(r) - \psi''(R)$$
$$= \left(\frac{r^{-s-1}+1}{8}\right)\left(\frac{r+1}{2}\right)^{s-2}$$
$$-\left(\frac{R^{-s-1}+1}{8}\right)\left(\frac{R+1}{2}\right)^{s-2},\ -1 \le s \le 2. \tag{49}$$

Again, from (47) we have

$$|\psi_s'''(x)| = \frac{1}{2(x+1)^3}\left(\frac{x+1}{2}\right)^s \times \tag{50}$$

$$\times\left[3x^{-s-1} + (s+1)x^{-2-s} + (2-s),\right]\ \ x \in (0,\infty)\ \ -1 \le s \le 2.$$

This gives

$$|\psi_s'''(x)| = -\frac{x^{1-s}}{2(x+1)^4} \times \tag{51}$$
$$\times\{x^{1-s}[12x^2 + 8(s+1)s + (s+1)(s+2) + x^4(s-2)(s-3)\}$$
$$\le 0, -1 \le s \le 2.$$

In view of (51), we can say that the function $|\psi_s'''|$ is monotonically decreasing in $x \in (0,\infty)$ *for* $-1 \le s \le 2$, and hence, for all $x \in [r,\ R]$, we have

$$\|\psi_s'''\|_\infty = \sup_{x\in[r,\ R]} |\psi_s'''(x)| \tag{52}$$
$$= \frac{1}{2(r+1)^2}\left(\frac{r+1}{2}\right)^2 \left[3r^{-s-1} + (s+1)r^{-2-s} + (2-s)\right],\ \ -1 \le s \le 2.$$

By applying the Theorem 5 of Taneja [8] along with the expression (49) and (52) for the measure (19) we get the first two parts of the bounds (38) and (39). The last part of the bounds (38) and (39) follows in view of (46) and Theorem 5 of Taneja [8].

In particular when $s = -1, 0, 1$ and 2, we get the result studied in Taneja [6, 7].

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